Thermoelasticity

Outline

- Heat Conduction Equation
- General 3-D Formulation
- Combined Plane Hooke's Law
- Stress Compatibility and Airy Stress Function
- Displacement

Heat Conduction Equation

- For zero heat sources and steady state, the heat onduction becomes Laplace equation

$$\nabla^2 T = 0.$$

- With appropriate thermal BCs, i.e. specified temperature or heat flux, the temperature field can be determined independent of the stress-field calculations.
- Once the temperature is obtained, elastic stress analysis procedures can then be employed to complete the problem solution.
- For us, the temperature distribution is usually a given condition.

General Formulation of Thermoelasticity

Formulation of Thermoelasticity – 2D

- **Plane stress** thermoelastic Hooke's law

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- Solution of the Airy Stress Function

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Stress Function Formulation without Body Forces

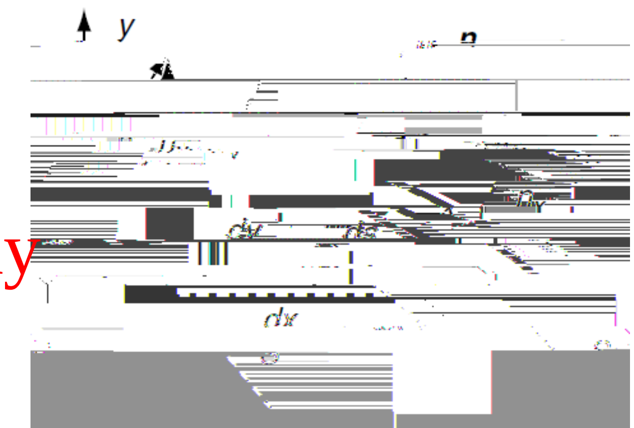
- Consider the directional derivative of the Airy Stress Function along the boundary normal

$$\frac{d}{d} \quad \text{—} n_x \quad \text{—} n_y \quad \left(\int_C T_y^n ds \right) \frac{dy}{d} \quad \left(\int_C T_x^n ds \right) \frac{dx}{d}$$

- where ***t*** is the unit tangent vector and ***F*** is the resultant boundary force.
- For many applications, the BCs are simply expressed in terms of stresses.
- For the case of zero surface tractions:

$$dy/dn = 0 \quad \text{Ÿ} \quad y = C.$$

- For simply connected regions, a steady temperature distribution with zero boundary tractions will not affect the in-plane stress field.



$$x \quad \frac{dx}{d} \quad \frac{dy}{d}, \quad y \quad \frac{dy}{d} \quad \frac{dx}{d}$$

Displacement Formulation – 2D

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Displacement Formulation – 2D

- Stress/Traction Boundary Conditions

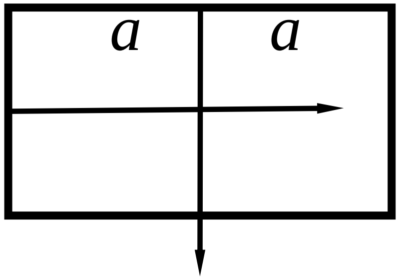
- Displacement Boundary Conditions

$$u = u_b(x,y)\,,\; v = v_b(x,y) \text{ on } S_u$$

Displacement Potential Formulation – 2D

- Navier's (governing) equations without body forces

Thermal Stresses in an Elastic Thin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- For $a >> b$, we may again ask Saint-Venant for help.
- Replace the parabolic surface traction with an equivalent (uniformly distributed) surface load.
- As a result, **the homogeneous Airy Stress Function**

$$\psi = cy^2 \Rightarrow \sigma_x'' = \frac{\partial^2 \psi}{\partial y^2} = 2c, \quad \sigma_y'' = \frac{\partial^2 \psi}{\partial x^2} = 0, \quad \tau_{xy}'' = -\frac{\partial^2 \psi}{\partial x \partial y} = 0$$

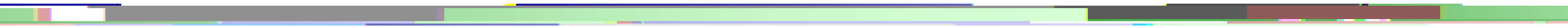
- Total stresses become

$$\sigma_x = \sigma_x' + \sigma_x'' = 2c - E\alpha T_0\left(1 - \frac{y^2}{b^2}\right), \quad \sigma_y = \sigma_y' + \sigma_y'' = 0, \quad \tau_{xy} = \tau_{xy}' + \tau_{xy}'' = 0$$

- Determine the temperature variation from heat conduction and energy equation, if not given.
- Under either stress or displacement formulation, identify a particular solution due to temperature effects to the governing equation (the Beltrami-

Polar Coordinate

- Strain-Displacement



Coordinate Formulation

[illegible]neous solution

Polar Coordinate – Displacement Formulation

- Axi-symmetric solution: $T = T(r)$

Thermal Stresses in an Annular Circular Plate

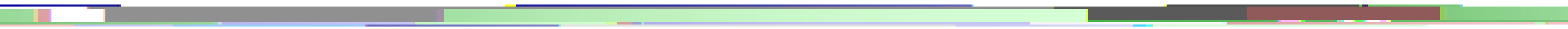
- After dropping the logarithmic term, the stress formulation gives

$$\sigma_r = \frac{C_3}{r^2} + C_2 - \frac{E\alpha}{r^2}\int Tr\,\mathrm{d}r, \quad \sigma_\theta = \frac{\mathrm{d}}{\mathrm{d}r}(r\sigma_r)$$

- Zero tractions on boundaries

- The displacement solution is

$$u_r \; (1 \;) \int_{r_i}^{r} T(\;)\,\mathrm{d} \quad \frac{(1 \;)r^2 \quad (1 \;)r_i^2}{\;^2_o \quad\; ^2_i} \int_{r_i}^{r_o} T(\;)\,\mathrm{d} \; .$$



Outline

- Heat Conduction Equation
- General 3-D Formulation
- Combined Plane Hooke's Law
- Stress Compatibility and Airy Stress Function
- Displacement Equilibrium and Displacement Potentials
- Thermal Stresses in Thin-Plates
- Summary of Solution Strategy
- Polar Coordinate: Airy Stress Function
- Polar Coordinate: Displacement Potentials
- Axi-symmetric Problems – Direct Solution